The evolution of a continuous projection for portraying the area of territorial units in proportion to a thematic variable has, in the past, eluded the cartographer. The present paper—resulting from postgraduate research in cartography by the second author under the supervision of the first—describes the development of such a projection and some of the possible applications.

A Polyfocal Projection for Statistical Surfaces

Naftali Kadmon and Eli Shlomi
Tel-Aviv University and Survey of Israel

A mathematical model is a conceptual abstraction, representing a generally complex situation in a quantitatively simplified way which may often be regarded as a streamlining of the actual data. Such a model may be quite generalised, and it can be modified chiefly in two ways. Either the general or overall parameters of the representation formulae can be changed, resulting in an improved version. Alternatively, 'local' modifications can be introduced. The representation of the earth with the aid of a cartographic projection is such a model. The actual shape of the earth with its mountains and valleys is, of course, quite irregular. As a first approximation the sphere with its single parameter—the radius—is generally used, and this serves as the basis for the computation of many 'geographical' map projections. Geodetic projections used for the production of large-scale topographic maps require a higher grade of precision in representing the earth's shape, and an ellipsoid of revolution therefore serves as a second approximation. Here the original spherical model is modified by introducing a second parameter and utilising the major and minor semi-axes for the definition of the spheroidal surface, still adhering to a regular mathematical shape. However, many large-scale geodetic applications require an even greater degree of adherence to the actual shape of the earth, and for the determination of altitudes a third approximation is used, namely the geoid. This equipotential surface can be regarded as having local deformations—albeit not necessarily mathematically defined ones—superimposed on a theoretical regular surface.

All cartographic projections used in mapping—both those of a precise geodetic nature in which a knowledge of deformations is of importance, as well as the more general or global projections—regard the earth as a regular surface. Deformations here vary as a function of the distance from given points or lines. We thus arrive at more or less regular patterns of deformations. One method of producing a projection with a better 'cling' to the sphere or the ellipsoid is arrived at by dividing the projection surface into regular strips, along standard lines, as is the case in the transverse cylindrical representation based on the Gauss–Krüger or the Transverse Mercator projection (*e.g.* the 60 zones of the Universal Transverse Mercator projection and UTM topographic grid system). Another is the modification of such projections as the Sanson–Flamsteed Sinusoidal by centering various portions on different half meridians, resulting in large gaps which are usually located in the oceans where the great, and sometimes infinite, distortions do the least damage, so to speak. As is true of all projections, their application must thus be judged not by 'absolute' merit but in conjunction with both the actual region covered and, chiefly, the purpose of the map.

RADIALLY-CHANGING SCALE PROJECTIONS

The modified plane projections cited above have one aim in common: to locally enhance certain properties such as conformality in an equivalent projection (*e.g.* interrupted Sinusoidal), or linear scale and area equivalence in a conformal projection (*e.g.* UTM). However, in the past a need has been felt for a cartographic representation which stresses, and even markedly exaggerates, some property such as linear scale or area around a given point or a number of points. The classic example is the suggestion for a logarithmic-scale map[1] showing emigration from Sweden generally attributed to Hägerstrand.[2] The parabolic-scale town maps by Falk are another, devised with the aim of showing the town centre at a larger scale than the periphery while retaining continuity (in contrast to the interrupted examples mentioned above). These parabolic maps and their 'projections' are, however, produced by mechanical means. A projection for producing town and road maps with scale changing radially at a hyperbolical rate from a central point to the periphery has been proposed by Kadmon.[3,4] This not only enables production of the map on an automated plotter with a program named HYPERBOMAP, but it makes it easy to change the projection centre as well as central scale and radial scale change, thus resulting in a family of hyperbolical-scale projections.

The above-mentioned projections can be regarded as stretching the map, fish-eye fashion, around a single point. It should indeed be stressed that in all cases a single central point or focus is used. Haggett even went so far as stating,[5] 'Constructing the same kind of map for two centres is possible only if we are prepared to drop the true-direction constraint. For three or more centres, no two-dimensional mapping is possible.'

POLYFOCAL CARTOGRAPHY

We shall now present a map projection which deforms the surface locally not at one point but at an arbitrary number of points. We must start with some observations on the possible application of such a map. In speaking of 'straight-lined' computer maps Kadmon suggested[6] that these can be used in the field of thematic mapping but not as topographic maps, much as a photographic image of the human body is required in an anatomical atlas, whereas a streamlined version is often found in art. The same is true of locally deformed (or rather locally deforming) projections. They can be utilised in portraying subjects in which not the 'true' or orthogonal outlines are of importance, but a locally deformed or induced modification in which scale at any point is proportional to some quantitative variable. Analog (*i.e.* physical) models of such projections produced by magnetic forces as well as by surface tension are shown by Morgan in Chorley and Haggett.[7]

The proper method for developing such a projection would be to 'graft' the local modification directly onto the spheroid and thence projecting the new surface into the plane with the aid of suitable transformation formulae. However, the authors, one of whom is a cartographer and the other a geodesist, were too lazy (let truth be said) to take the long and theoretically precise way. Instead, they set out from a given plane representation of the earth, *i.e.* from a map, by digitising it with all its inherent limitations and deformations, arriving at the final product by subjecting this base map to the transformations developed below. For these, the name Polyfocal Projection is suggested, conforming to naming by 'special property'—attribute (c) as listed by Maling.[8] The authors admit *a priori* the shortcomings of this method from a theoretical viewpoint and thus fend off any criticism raised by purists enquiring after a precise definition of the deformations incurred in this projection. However, if we know the properties of the original base map projection, such as the Tissot indicatrix distribution, the polyfocal projection can be superimposed on these and we would arrive at a reasonably 'true' modification of the spheroid. In practice an existing world map was used, as digitised by Mr G. B. Lewis, to whom our thanks are due, based on the van der Grinten projection. This file of points was then subjected to the transformations of the polyfocal projection.

The development of a new projection—and the authors hope that the work outlined in this article merits this classification (see Maling's note on 'Personal Projections'[9])—may be interesting and rewarding in itself, whether there is or is not an actual need for it. But most projections evolve, after all, with a practical application in mind. We shall, therefore, briefly review some fields in which the polyfocal projection might be utilised. Others might be devised, and if they are of any practical use, cartographers will find their own purpose for trying them out.

The first use would conceivably be in the economic field. One often finds map series showing different territorial units (*e.g.* countries) with area proportional to some statistical variable, such as fuel oil production or consumption. These are generally made by redrawing each country separately at the proper thematic scale, and then fitting the countries together as well as practicable. This not only introduces deformations, but requires the successive repositioning of the countries in a mostly discontinuous array. Hand-drawn or one-off discontinuous maps with locally changing scale can be found in a wide range of publications, from serious atlases[10] through textbooks[11] and professional journals[12] to popular periodicals.[13] This manual redrawing is a tedious job, and might be replaced by using the polyfocal projection; the result will be a suitably deformed but continuous representation of the statistical surface. Another application might be in the domain of road and communication maps. In Kadmon (1974) it was suggested that the hyperbolic-scale town maps described could be used for portraying travel *time* on a linear scale. These maps were based on a single centre on the assumption that traffic density and average speed change radially from this point. However, in order to better approximate real conditions it would be necessary to re-centre these projections at a number of foci simultaneously. The present projection was, in fact, developed with this problem in mind. Another application could be in the field of mapping potentials, since the definition of potential at a point as the minus first moment of the population about this point (see *e.g.*[14]) somewhat resembles the basic expression used in developing the polyfocal projection.

In mapping a statistical surface, continuous or with discontinuities such as choropleths, one generally superimposes a scale of tones or tints over the geometrically 'true' representation of the territorial units in order to express thematic value or intensity. It is then difficult to graft further continuous variables onto this image. If, instead of 'flattening' the statistical surface orthogonally into the map plane, we express the shapes in a form which is proportional in size to the thematic variable, we can add another thematic dimension with the aid of colour, thus enabling in a single map the comparison of two different area variables such as production (size of area) and consumption (colour).

Various geographers have tried in the past to devise a cartographic projection which would fulfil the needs for a continuous representation of non-Euclidean or non-metric spaces, *i.e.* of spaces in which distance or area are proportional to non-geometric variables such as cost, time, density *etc.* Tobler reproduces the Armadillo 'projection' presented (drawn by freehand) by Woytinsky (see Tobler, 1963, p. 62). Bunge[15] extensively treats such spaces. But apparently no one has, as yet, arrived at a real projection of this type.

One of the more challenging applications of the polyfocal projection is now being tested. Cristaller, in his Theory of Central Places,[16] covers thematic space with a net of hexagons of equal thematic content but different geometrical area. Bunge[15] (p. 278) tried to convert this into a space composed of hexagons of uniform density, *i.e.* of equal area, but succeeds in doing this only approximately and subjectively by freehand. The polyfocal projection now can do this objectively by a method of iterations. *Figure 5* shows Bunge's original Cristaller hexagons, while *Figure 6* shows their equal-area (and therefore equal-density) transformation with the aid of the polyfocal projection.

DEVELOPING THE PROJECTION

If $S_0 = 1/s_0$ is the original scale of the map, R is the distance of a point P from the focus in the original projection, and $f(R)$ is the distance function (called distance friction by some writers; see *e.g.*[17]) which describes the effect of distance on the intensity of the phenomenon (and therefore on map scale at point P), then scale S in a radial direction over the distance R in the new projection may be given by

$$S = S_0 + S_0 \cdot f(R) \tag{1}$$

Preferably S should (i) decrease with increasing R, (ii) be finite and continuous at the focus, where $R = 0$. The following form can thus be suggested for $f(R)$

$$f(R) = \frac{a}{b + c \cdot R^{(d_0 + d_1 t + d_2 t^2 + \ldots + e_1 u + e_2 u^2 + \ldots)}} \tag{2}$$

where a, b, c, d_i and e_i are empirical parameters representing factors influencing the phenomenon. Thus, t and u could represent time and cost, influencing scale/distance relations. However, we shall limit ourselves to functions of a single variable.

By dividing by one of the parameters, say b, this expression (2) can be simplified thus

$$f(R) = \frac{a/b}{b/b + (c/b) \cdot R^{(d_0 + d_1 t + d_2 t^2 + \ldots)}}$$

and if $a/b = A$, $c/b = C$,

$$f(R) = \frac{A}{1 + C \cdot R^{(d_0 + d_1 t + d_2 t^2 + \ldots)}} \tag{3}$$

We shall now assume that the influence of the focus decreases simply with the square of the distance, *i.e.* $d_0 = 2$ and $d_i \equiv 0$ ($i = 1, 2, 3, \ldots$). The resulting distance friction function

$$f(R) = \frac{A}{1 + C \cdot R^2} \tag{4}$$

thus has a form similar to that of every term of the function of potential

$$P_j = \sum_i \frac{M_i}{R_{ij}^2}$$

(see *e.g.* Ref. 14). The latter, though, is discontinuous when $R = 0$, whereas the present function is fully defined at the focus. Substituting (4) in (1) and assuming $S_0 = 1$ (tantamount to changing the projection of an existing map) we have

$$S = 1 + \frac{A}{1 + C \cdot R^2} \tag{5}$$

Since scale is to change continuously and radially from the focus,

$$r = S \cdot R$$

where r is the radial distance after transformation, corresponding to original distance R. The former, in terms of the original coordinate net, is thus a function of the original distance value and the parameters of the transformation function, *i.e.*

$$r = R + \frac{A \cdot R}{1 + C \cdot R^2} \tag{6}$$

A measures the 'power' of the focus, *i.e.* the absolute value of the thematic variable at this point. C denotes the radial rate of change of this variable.

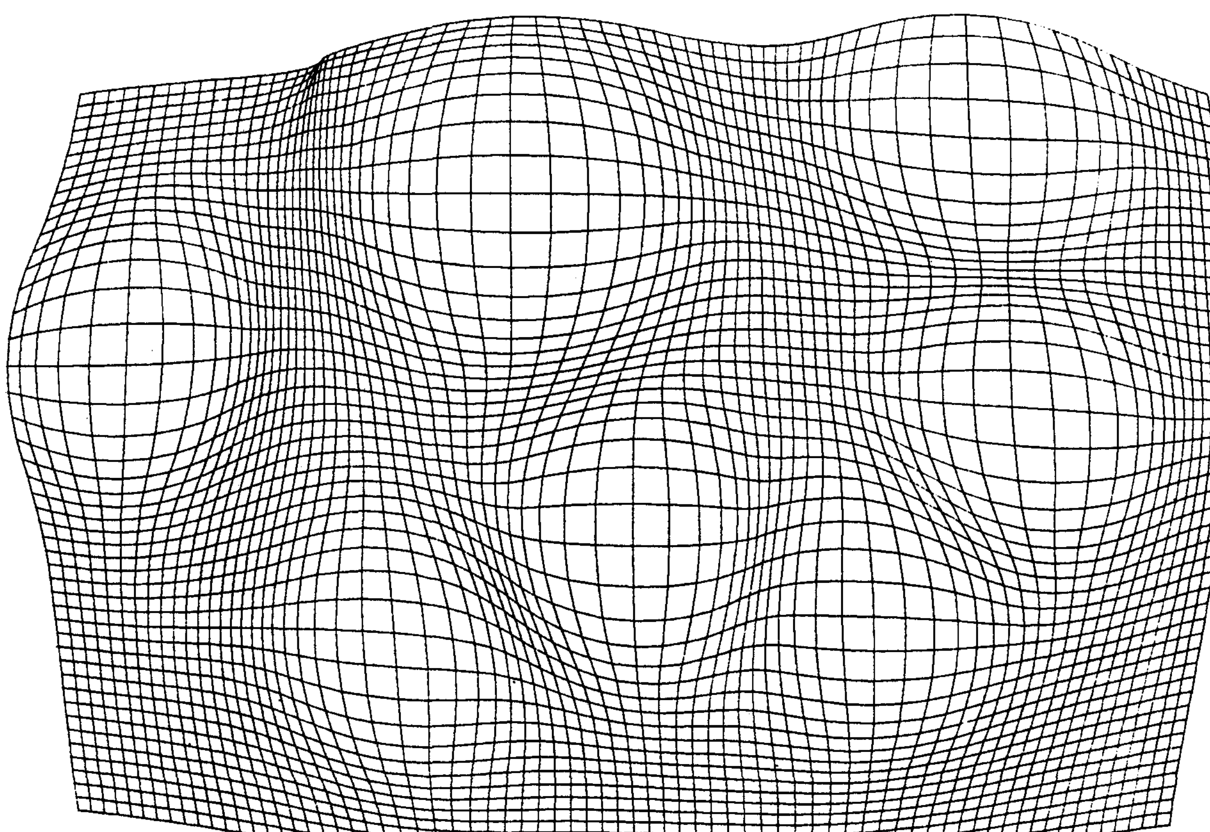


Figure 1. The polyfocal transformation of a square grid. The foci with their surrounding area of enlarged scale can easily be discerned.

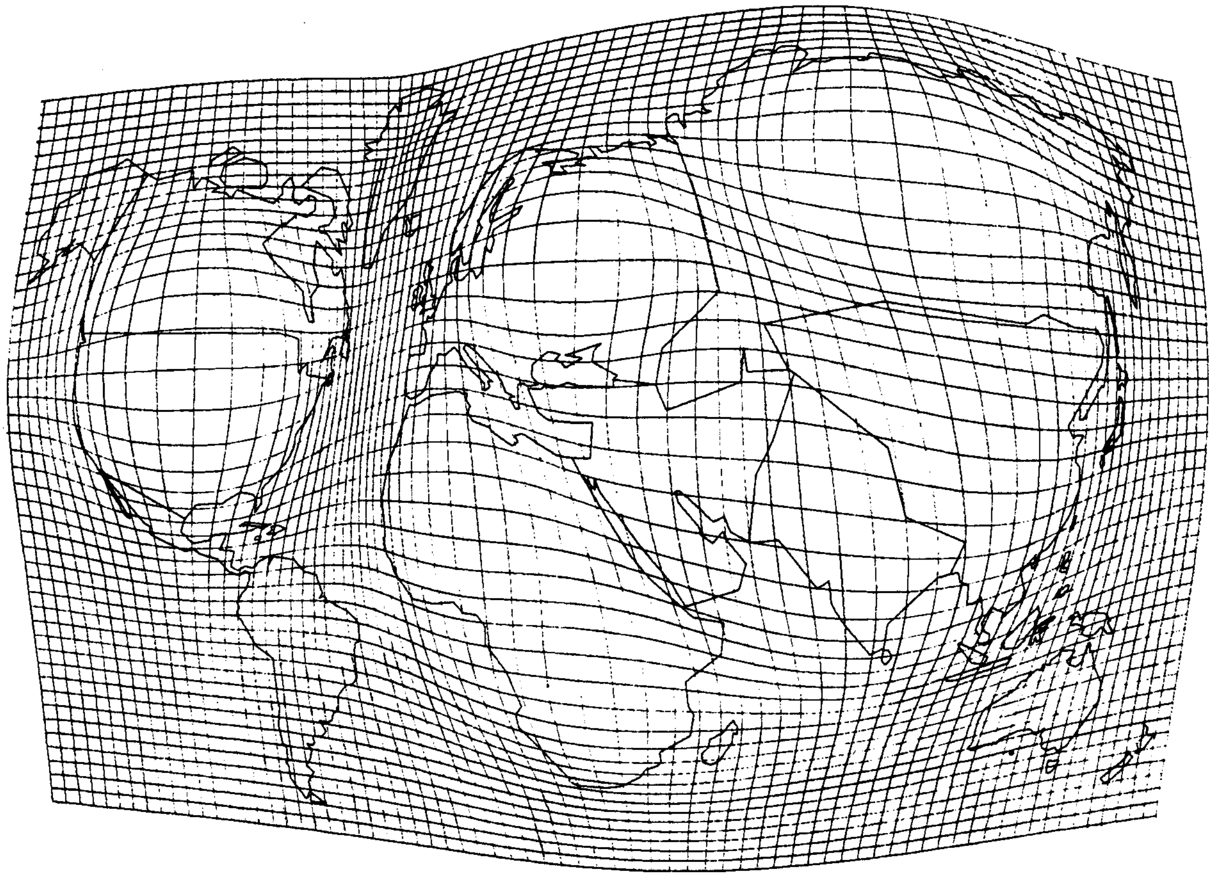


Figure 2. A polyfocal world map with division into main blocks, each enlarged or reduced in scale in proportion to a thematic variable.

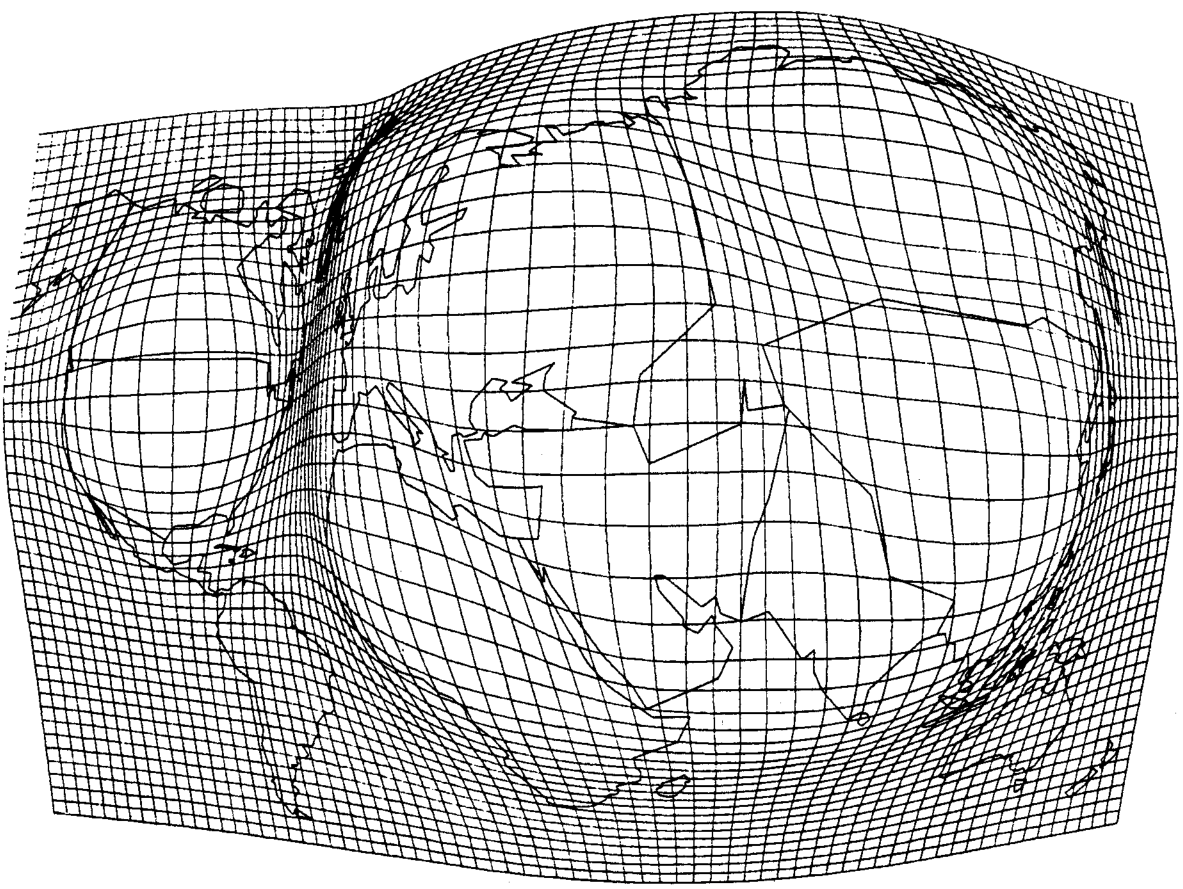


Figure 3. The same map after a change of local scale parameters.

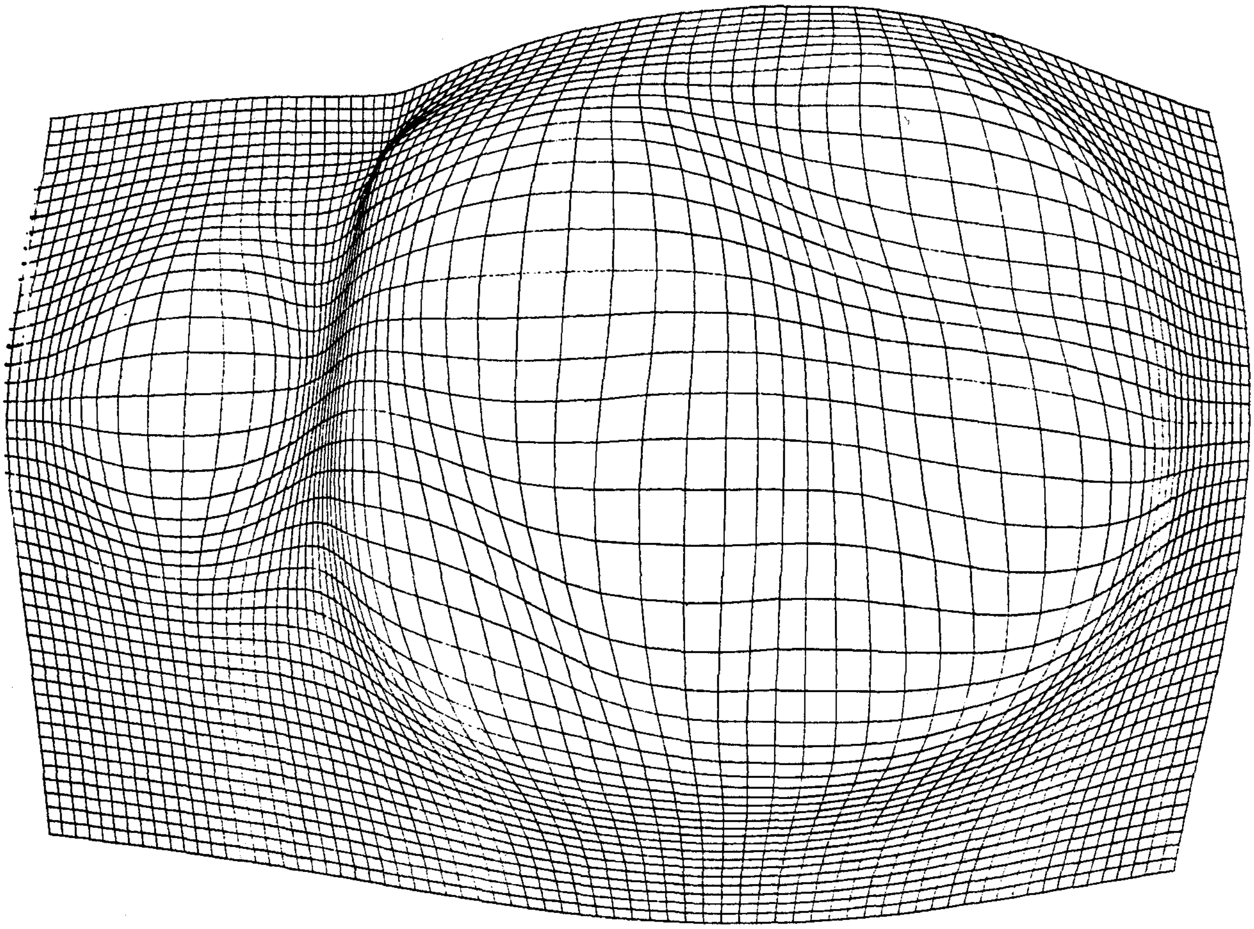


Figure 4. The grid of the map shown in *Figure 3*. A 'vanishing area' of zero scale can be seen in the North Atlantic. Even negative scales can be envisaged.

If X_1, Y_1 are the coordinates of the focus, then the radial distance R of point P from this focus is

$$R = \sqrt{(x-X_1)^2+(y-Y_1)^2}$$

In the new projection the respective distance is represented by (6), and, since the original coordinates of point P are

$$\left.\begin{aligned} x &= X_1+R\cos\alpha = X_1+dx \\ y &= Y_1+R\sin\alpha = Y_1+dy \end{aligned}\right\} \quad (7)$$

the rectangular ('new') coordinates of point P' corresponding to P in the original map (assuming azimuthality at the focus) are

$$\left.\begin{aligned} x' &= X_1+r\cos\alpha \\ y' &= Y_1+r\sin\alpha \end{aligned}\right\} \quad (8)$$

Substituting (6) in (8),

$$x' = X_1+\left(R+\frac{A\cdot R}{1+C\cdot R^2}\right)\cdot\cos\alpha$$

$$y' = Y_1+\left(R+\frac{A\cdot R}{1+C\cdot R^2}\right)\cdot\sin\alpha$$

or

$$\left.\begin{aligned} x' &= X_1+R\cos\alpha+\frac{A\cdot R\cos\alpha}{1+C\cdot R^2} \\ y' &= Y_1+R\sin\alpha+\frac{A\cdot R\sin\alpha}{1+C\cdot R^2} \end{aligned}\right| \quad (9)$$

However, from (7)

$$R\cos\alpha = x-X_1 = dx$$

$$R\sin\alpha = y-Y_1 = dy$$

so that

$$\left.\begin{aligned} x' &= x+\frac{A\cdot dx}{1+C\cdot R^2} = x+\Delta x \\ y' &= y+\frac{A\cdot dy}{1+C\cdot R^2} = y+\Delta y \end{aligned}\right| \quad (10)$$

With the aid of this transformation the entire original map space can be re-mapped around a single focus with scale tending to 1 (*i.e.* to original scale) as radial distance increases.

THE MULTIFOCAL MAP

In order to portray a spatial distribution influenced by a number of foci it will be assumed that the value of the thematic variable at any point is equal to the sum of the values induced by the individual foci. The new coordinates of a point will then be equal to the original coordinates plus the values computed from the individual foci.

In (10) we saw that the coordinates of point P' are composed of (i) the original coordinates of point P, (ii) an increment which is a function of the radial distance from the focus and the parameters of the latter. For n foci we shall now write

$$
\left.\begin{aligned}
x' &= x+\Delta x_1+\Delta x_2+\ldots+\Delta x_n = x+\sum_{i=1}^{n}\Delta x_i \\
y' &= y+\Delta y_1+\Delta y_2+\ldots+\Delta y_n = y+\sum_{i=1}^{n}\Delta y_i
\end{aligned}\right\} \quad (11)
$$

For n foci (X_i, Y_i), $(i = 1, 2, \ldots, n)$ with parameters A_i, C_i, the equations of the new projection, derived from the original map and the distances R_i of a point (x, y) from the foci in the latter, are

$$
\left.\begin{aligned}
x' &= x+\sum_{i=1}^{n}\frac{A_i\cdot(x-X_i)}{1+C_i\cdot R_i^2} \\
y' &= y+\sum_{i=1}^{n}\frac{A_i\cdot(y-Y_i)}{1+C_i\cdot R_i^2}
\end{aligned}\right\} \quad (12)
$$

where $R_i = \sqrt{(x-X_i)^2+(y-Y_i)^2}$

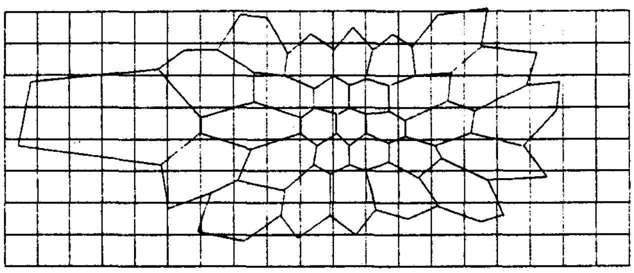
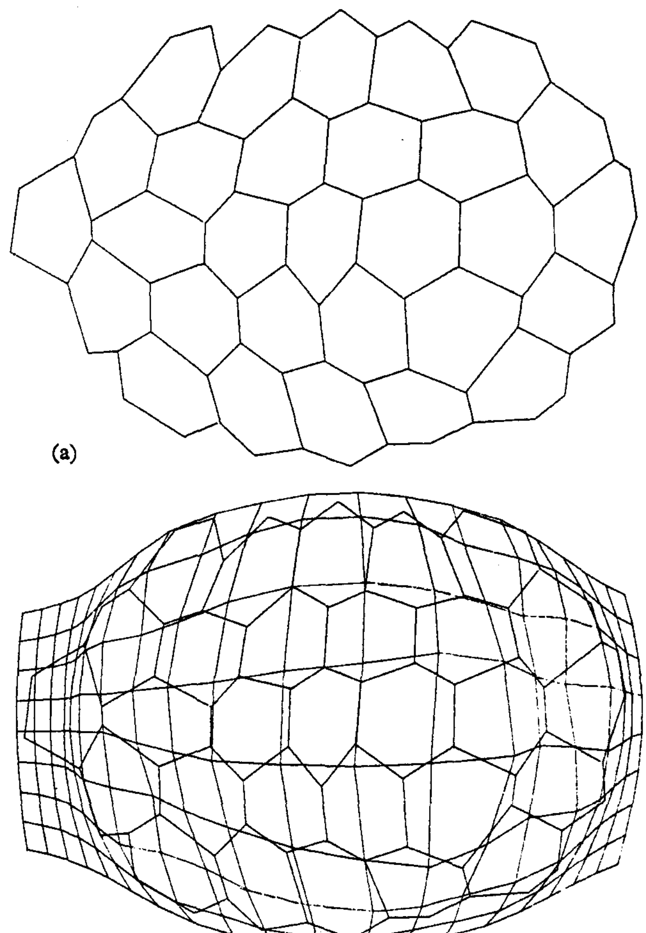


Figure 5. Cristaller hexagons with equal thematic content on a square grid (after Bunge).

PROGRAM POLYMAP

The Polyfocal Projection would be another case (perhaps it is!) of drawer cartography—mapping innovations which are kept in a drawer—if it had not been computerised from the start. A conventional projection can be adapted to a specific use simply by 'photographic' change of scale. The whole *raison d'être* of the Polyfocal Projection is its application to occurrences where no two cases are geometrically similar. A FORTRAN program named POLYMAP has been written to produce such maps on a Calcomp 11 inch drum plotter. Chief variables are 'basic' scale and a file of points (the foci) at which scale changes are induced, as well as the weights or influences of these points A_i and the rates of change C_i of this influence. These are followed by the point coordinates of the original map. The addition of a rectangular grid serves not only to stress the form of local elasticity or deformations, but also as a measure of scale change and therefore of local thematic value. Some of the preliminary results are shown in the accompanying figures.



(a)

(b)

Figure 6. The same hexagons after being transformed by the polyfocal projection into units of approximately equal density (*i.e.* nearly equal area), (a) without grid, (b) with transformed grid. Any specified degree of precision can be obtained by increasing the number of iterations.

References

1. W. R. Tobler (1963). Geographical area and map projections, *The Geographical Review*, **53**, 66.
2. R. Abler, J. Adams and P. Gould (1971). *Spatial Organization.* Prentice-Hall. These authors (as well as others) state (p. 77) that the logarithmic projection was suggested to Hägerstrand by Edgar Kant.
3. N. Kadmon (1974). A data-bank derived hyperbolic-scale town map series. Paper presented to the Seventh International Conference on Cartography, Madrid.
4. N. Kadmon (1975). Data-bank derived hyperbolic-scale equitemporal town maps, *International Yearbook of Cartography*, **15**, 47–54.
5. P. Haggett (1972). *Geography—A Modern Synthesis.* Harper and Row, p. 98.
6. N. Kadmon (1971). KOMPLOT—do-it-yourself computer cartography, *The Cartographic Journal*, **8** (2), 141.
7. M. A. Morgan (1971). Hardware models in geography. In Chorley and Haggett, *Models in Geography*. Methuen & Co., pp. 727–70.
8. D. H. Maling (1973). *Coordinate Systems and Map Projections.* George Philip and Son, p. 108.
9. D. H. Maling (1974). Personal projections, *Geographical Magazine*, August 1974, pp. 559–60.
10. *The TIMES Concise Atlas of the World* (1972). TIMES Newspapers and John Bartholomew and Son Ltd., Edinburgh. See *e.g.* the maps on pp. 21 and 29.
11. R. Cuenin (1972). *Cartographie Generale.* Tome 1, Notions generales et principes d'elaborations. Edition Eyrolles, Paris, p. 310.
12. J. M. Hunter and M. S. Meade (1971). Population models in the high school, *Journal of Geography*, **70** (2), 101.
13. Das Milliarden-Spiel. *Stern*, no. 51/1975, 11 Dec. 1975, p. 49.
14. W. Warntz and D. Neft (1960). Contributions to a statistical methodology for area distributions, *Journ. Regional Science Assoc.*, 2.
15. W. Bunge (1966). *Theoretical Geography.* Lund series in geography; Series C, general and mathematical geography. Gleerup, Lund. pp. 278–9.
16. W. Cristaller (1933). *Die zentralen Orte in Süddeutschland.* Gustav Fischer, Jena.
17. D. L. Huff and G. F. Jenks (1968). A graphic interpretation of the friction of distance in gravity models, *Annals, AAG*, **58** (4) (Dec. 1968), pp. 814–24.
18. E. Shlomi (1977). A polyfocal projection—development and applications. Unpublished M.A. thesis, Tel-Aviv University.